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On the convergence of the collocation method for nonlinear boundary integral equations

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Abstract

Recently, Galerkin and collocation methods have been analysed for some nonlinear boundary integral equations. For the collocation method it has been assumed that the nonlinearity is asymptotically linear. In this paper we remove this restriction. We shall prove the convergence of the collocation method for nonlinear boundary integral equations, when the nonlinearity has a polynomial growth condition. In addition to this the optimal order error estimates follow in $L^q(\Gamma)$-norm.

Key words: Boundary element method; Nonlinear; Elliptic; Boundary value problem; Galerkin method; Collocation method; Potential problems; Nonlinear material conditions

1. Introduction

In this paper we study the boundary element collocation method applied to a nonlinear elliptic boundary value problem:

$$\begin{cases} \Delta\phi = 0, & \text{in } \Omega, \\ -\left.\dfrac{\partial\phi}{\partial n}\right|_{\Gamma} = G(\phi|_{\Gamma}) - f, & \text{on } \Gamma. \end{cases} \tag{1.1}$$

The boundary data f has been given in some appropriate function space. Here $\Omega \subset \mathbb{R}^2$ is a bounded plane domain with a smooth boundary Γ. This kind of boundary value problems is encountered in a stationary heat conduction problem with a nonlinear radiation law across the boundary [3], or generally in diffusion processes, where the "fluid flow" across the boundary depends also on the solution inside the domain.

The indirect boundary integral formulation is applied to problem (1.1). As in the linear case, it is assumed that the harmonic function ϕ has the representation

$$\phi(x) = -\frac{1}{2\pi}\int_{\Gamma} u(y) \log|x-y| \, \mathrm{d}s_y, \quad x \in \Omega. \tag{1.2}$$

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Assume for the moment that the boundary distribution u has been found. Using the trace properties (i.e., jump relations) of the single-layer potential, we notice that u solves the nonlinear boundary integral equation

$$\tfrac{1}{2}u - D^*u + G(Su) = f, \tag{1.3}$$

where D^* is the spatial adjoint of the double-layer operator D:

$$Du(x) = \frac{1}{2\pi}\int_\Gamma(y)\frac{\partial}{\partial n_y}\log|x-y|\,\mathrm{d}s_y, \quad x\in\Gamma,$$

and S is Symm's integral operator of the first kind:

$$Su(x) = -\frac{1}{2\pi}\int_\Gamma u(y)\log|x-y|\,\mathrm{d}s_y, \quad x\in\Gamma.$$

In the first part of the paper we shall briefly recall the solvability of (1.3) from [12]. For this we have to make some additional assumption concerning the nonlinearity $G(\cdot)$.

The main focus, however, is to analyze the collocation methods used to solve (1.3) numerically. In [6,9,13,14] it has been frequently assumed that the nonlinearity $G(\cdot)$ is asymptotically linear. Applying some strong results of the nonlinear functional analysis (methods of a-proper mappings and related techniques) this assumption may be removed. We allow the nonlinearity to have at most a polynomial growth with the leading exponent $p\geqslant 1$.

With the theory of a-proper mappings we finally study the discretized equations. We prove the existence of the approximate solutions and the convergence of these methods.

2. Formulation of the problem: preliminaries

Let us first describe the function spaces which shall be used throughout the paper. In what follows let Ω be a bounded domain in $\mathbb{R}^2$ with a smooth boundary. This means that the boundary $\Gamma=\partial\Omega$ has a regular parameter representation $x:\mathbb{R}\to\Gamma$ which is one-periodic and the Jacobian $|\mathrm{d}x/\mathrm{d}t|\neq 0$.

For every $m\in\mathbb{N}$ and $1<p<\infty$ we denote by $W^{m,p}(\Omega)$ the Sobolev space with the usual topology induced by the norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{m,p,\Omega} = \left\{\sum_{|\alpha|\leqslant m}\|D^\alpha u\|^p_{L^p(\Omega)}\right\}^{1/P}, \tag{2.1}$$

where $D^\alpha=(\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}$, $|\alpha|=\alpha_1+\alpha_2$.

The Sobolev–Slobodetckii space $W^{s,p}(\Gamma)$, $s\geqslant 0$, on the boundary is defined as a completion of $C^\infty(\Gamma)$ with respect to the norm

$$\|u\|_{s,p,\Gamma} = \left\{\|u\|^p_{[s],p,\Gamma} + \int_\Gamma\int_\Gamma\frac{|D^{[s]}u(x)-D^{[s]}u(y)|^p}{|x-y|^{1+p(s-[s])}}\mathrm{d}s_x\,\mathrm{d}s_y\right\}^{1/p}.$$

Here we used the notation $s=[s]+\delta$, where $[s]\in\mathbb{N}$ is the largest natural number smaller than s, and $0\leqslant\delta<1$. The negative order Sobolev–Slobodetckii spaces are defined by duality with respect to the $L^2(\Gamma)$-inner product (see, e.g., [1]).

Next we shall formulate the problem (1.1) properly in mathematical terms.

The nonlinear boundary value problem. Find $\phi \in W^{1,2}(\Omega)$ such that

$$\begin{cases} \Delta\phi = 0, & \text{in } \Omega, \\ -\dfrac{\partial\phi}{\partial n} = G(\phi) - f, & \text{on } \Gamma, \end{cases} \tag{2.2}$$

where $f \in W^{-1/2,p}(\Gamma)$. The boundary condition is satisfied in the $W^{-1/2,p}(\Gamma)$-sense.

The nonlinear term $G(\cdot)$ acting on the boundary is supposed to be a Nemitskyi operator corresponding to a Caratheodory function $g(x, u): \Gamma \times \mathbb{R} \to \mathbb{R}$, which by the definition of the Caratheodory function is measurable in x for almost all $u \in \mathbb{R}$, and continuous in u for almost all $x \in \Gamma$. We shall assume that the Caratheodory function is such that the following assumptions are valid.

Assumption A$_1$. *The nonlinear mapping* $G: L^p(\Gamma) \to L^q(\Gamma)$, $2 \leqslant p < \infty$, $1/p + 1/q = 1$, *is bounded, continuous and strictly monotone, i.e.,*

$$(G(u) - G(v), u - v) > 0, \quad \forall u \neq v \in L^p(\Gamma).$$

In addition to this we suppose that for almost all $x \in \Gamma$,

$$G(u)(x)u(x) \geqslant a\,|\,u(x)\,|^{p} + b, \quad a > 0,\, b \in \mathbb{R}, \tag{2.3}$$

$$|\,G(u)(x)\,| \leqslant \alpha\,|\,u(x)\,|^{p-1} + \beta, \quad \alpha > 0,\, \beta \in \mathbb{R}. \tag{2.4}$$

Now by [12] the following theorem holds.

Theorem 2.1. *Suppose that Assumption* A$_1$ *is valid. Then the potential problem* (2.2) *has a unique solution* $\phi \in W^{1,2}(\Omega)$ *for every* $f \in L^q(\Gamma)$.

3. The boundary integral equation

The nonlinear potential problem

$$\begin{cases} \Delta\phi = 0, & \text{in } \Omega, \\ -\dfrac{\partial\phi}{\partial n} = G(\psi) - f, & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{3.1}$$

will be formulated as a nonlinear boundary integral equation. We proceed here as in [12]. We introduce a boundary distribution u such that

$$\phi(x) = -\frac{1}{2\pi}\int_\Gamma u(y) \log|\,x - y\,|\, \mathrm{d}s_y, \quad x \in \Omega. \tag{3.2}$$

Using the trace properties of the normal derivative of the single-layer potential, we notice that the boundary distribution u satisfies the nonlinear boundary integral equation

$$A(u) = \left(\tfrac{1}{2}I - D^*\right)u + G(Su) = f, \tag{3.3}$$

where D^* and S are defined by

$$D^*u = \frac{1}{2\pi}\int_\Gamma u(y)\frac{\partial}{\partial n(x)}\log|x-y|\,\mathrm{d}s_y, \qquad Su = -\frac{1}{2\pi}\int_\Gamma u(y)\log|x-y|\,\mathrm{d}s_y.$$

In [12] it was shown that the integral equation formulation is equivalent to the variational problem of (3.1). Moreover, we were able to prove the unique solvability of the boundary integral equation (3.3). Here we shall briefly review the basic properties of the integral operator.

We start our considerations by recalling some mapping properties of the single-layer and double-layer operators (cf. [6]).

Theorem 3.1. *The integral operators S, D and D^* have the following mapping properties:*
(1) $S, D, D^* : L^p(\Gamma) \to L^q(\Gamma)$ *are bounded and compact for all* $p, q \in [1, \infty]$;
(2) $S, D, D^* : L^p(\Gamma) \to C(\Gamma)$ *are compact*;
(3) $S : W^{t,p}(\Gamma) \to W^{t+1,q}(\Gamma)$ *is an isomorphism for all* $t \in \mathbb{R}$, $1/p + 1/q = 1$, $1 < p, q < \infty$.

For the unique solvability the following statement is crucial [12, Theorem 5].

Theorem 3.2. *The integral operator $A : L^q(\Gamma) \to L^q(\Gamma)$ is a strictly S-monotone mapping, i.e., for all $u, v \in L^q(\Gamma)$,*

$$(A(u) - A(v), S(u-v))_{L^2(\Gamma)} > 0, \quad u \neq v.$$

By Assumption A_1 and the previous theorem we finally obtain, using the unique solvability of the corresponding variational problem [12, Theorem 7], the following theorem.

Theorem 3.3. *For every $f \in L^q(\Gamma)$ there exists a unique solution $u \in L^q(\Gamma)$ such that*

$$A(u) = (\tfrac{1}{2}I - D^*)u + G(Su) = f.$$

For the analysis of the numerical schemes we shall make two additional assumptions.

Assumption A_2. *The nonlinear operator $G : L^p(\Gamma) \to L^q(\Gamma)$ is continuously Fréchet differentiable and the Fréchet derivative $G'(u) : L^p(\Gamma) \to L^q(\Gamma)$ is a positive operator:*

$$(G'(u)h, h)_{L^2(\Gamma)} > 0.$$

Note that for the Nemitskyi operator the Fréchet derivative is a multiplicative operator such that for almost all $x \in \Gamma$,

$$G'(u)v(x) = \frac{\partial g}{\partial u}(x, u(x))v(x).$$

Furthermore, we assume that the nonlinear mapping is sufficiently smooth. It means that the restriction of the mapping $G(\cdot)$ on continuous functions has the following properties.

Assumption A_3. *$G: C(\Gamma) \to C(\Gamma)$ is bounded, and for every $x \in \Gamma$ and for all $u, v \in C(\Gamma)$ there exists constants $K_i > 0$, $i = 1, 2$, such that*

$$|G(u(x)) - G(v(x))| \leqslant |u(x) - v(x)| \left\{ K_1 + K_2 \left(|u(x)|^{p-2} + |v(x)|^{p-2} \right) \right\}.$$

In addition to this we suppose that the Fréchet derivative $G'(u): C(\Gamma) \to C(\Gamma)$ is bounded and continuous in u.

Now we have the following theorem [12, Theorem 9].

Theorem 3.4. *Suppose that Assumptions A_1–A_3 are valid. Then $A: L^q(\Gamma) \to L^q(\Gamma)$ is bounded, and continuously differentiable. Moreover, the Fréchet derivative $A'(u): L^p(\Gamma) \to L^q(\Gamma)$ is a Fredholm operator with index zero*: $\operatorname{ind}(A'(u)) = 0$. *In addition to this, the Fréchet derivative defined by*

$$A'(u)h = \left(\tfrac{1}{2}I - D^*\right)h + G'(Su)Sh$$

is also injective.

4. The boundary element method

In this section we shall study the collocation method used for solving numerically the boundary integral equation (3.3). For the definition of the approximation schemes we introduce the boundary element spaces. First we define the periodic splines on the unit interval [0, 1] [16]:

$$V_N^d(\Theta) = \left\{ \phi \in C^{d-1}(0, 1) \middle| \, \phi|_{I_j} \in P^d, \ j = 0, \dots, N-1, \ \phi^{(k)}(0) = \phi^{(k)}(1), \ 0 \leqslant k \leqslant d-1 \right\},$$

where $\Theta = \{0 = \tau_0 < \tau_1 < \cdots < \tau_N = 1\}$ is a partition of [0, 1] and $I_j =]\tau_j, \tau_{j+1}[$. Using the parameter representation $x: [0, 1] \to \tau$, we define

$$S_N^d(\Gamma)\{\chi \in C^{d-q}(\Gamma) \,|\, \chi(x(\cdot)) \in V_N^d(\Theta)\}.$$

We assume that the family of partitions are quasi uniform:

$$0 < \gamma \leqslant \frac{\max h_j}{\min h_i} = \frac{\max |x_{j+1} - x_j|}{\min |x_{i+1} - x_i|} \leqslant \gamma^{-1}.$$

For the approximation spaces the celebrated approximation and inverse properties hold [2,4,7,16].

Approximation property. For every $u \in W^{s,2}(\Gamma)$, $s \leqslant d+1$, there exists $\psi \in S_N^d(\Gamma)$ such that

$$\| u - \psi \|_{W^{t,2}(\Gamma)} \leqslant c h^{s-t} \| u \|_{W^{s,2}(\Gamma)},$$

$t \leqslant s$, $t < d + \frac{1}{2}$. The constant c depends only on s, t and d, but not on the mesh parameter $h = 1/N$.

Inverse property. For all $\psi \in S_N^d(\Gamma)$ there holds the inverse estimate

$$\|\psi\|_{W^{s,2}(\Gamma)} \leqslant ch^{t-s}\|\psi\|_{W^{t,2}(\Gamma)}, \quad t \leqslant s < d + \tfrac{1}{2}.$$

For the collocation method we define the interpolation points as follows:

$$x_i = x(\tau_i), \quad d = 2j+1, \qquad x_i = x\left(\tfrac{1}{2}\{\tau_i + \tau_{i+1}\}\right), \quad d = 2j,$$

where τ_i is the grid point of the partition Θ. The interpolation operator $I_h : C(\Gamma) \to S_N^d(\Theta)$ is defined by setting

$$I_h u(x_i) = u(x_i), \quad \forall i = 0, \dots, N-1. \tag{4.1}$$

Since the partitions are quasi uniform, the interpolation operators are uniformly bounded in $C(\Gamma)$ [4].

The collocation method consists in finding $u_h \in S_N^d(\Gamma)$ such that

$$I_h A(u_h) = I_h f, \tag{4.2}$$

or equivalently,

$$\left(\tfrac{1}{2}I - I_h D^*\right)u_h + I_h G(Su_h) = I_h f.$$

We shall use the theory of a-proper mappings (cf. [5,11]) to show that (4.2) has a unique solution provided that $h > 0$ is sufficiently small. First we have to verify the following theorem.

Theorem 4.1. *The operator* $A(\cdot) = \frac{1}{2}I - D^* + G(V(\cdot)) : L^q(\Gamma) \to L^q(\Gamma)$ *is a-proper with respect to the projectional scheme* $\pi = \{S_{N(h)}^d(\Theta),\ I_h\}$ *and a continuous right-hand side* $f \in C(\Gamma)$.

Proof. Suppose that $\{\phi_j\}$ is a bounded sequence in $L^q(\Gamma)$ such that $\phi_j \in S_{N_j}^d$ and

$$I_{h_j} A(\phi_j) = I_{h_j} f,$$

where $f \in C(\Gamma)$.

Since $L^q(\Gamma)$ is a reflexive Banach space, we can subtract a weakly converging subsequence. Let us denote this subsequence by $\{\phi_j\}$ and the weak limit by $v \in L^q(\Gamma)$.

The linear operators D^*, $S : L^q(\Gamma) \to C(\Gamma)$ are compact (Theorem 3.1). Hence we get

$$\lim_{j\to\infty} \|D^*(\phi_j - v)\|_{L^\infty(\Gamma)} = 0, \qquad \lim_{j\to\infty} \|V(\phi_j - v)\|_{L^\infty(\Gamma)} = 0. \tag{4.3}$$

Due to the quasi uniformity assumption the interpolation operators $I_h : C(\Gamma) \to L^q(\Gamma)$ are uniformly bounded independent of the mesh parameter h. This implies that

$$\begin{aligned}\|I_{h_j}D^*\phi_j - D^*v\|_{L^q(\Gamma)} &\leqslant \|I_{h_j}D^*\phi_j - I_{h_j}D^*v\|_{L^q(\Gamma)} + \|I_{h_j}D^*v - D^*v\|_{L^q(\Gamma)} \\ &\leqslant c(q)\|D^*\phi_j - D^*v\|_{L^\infty(\Gamma)} + \|I_{hj}D^*v - D^*v\|_{L^q(\Gamma)}.\end{aligned}$$

By (4.3) and the approximation properties of the spline spaces the right-hand side tends to zero as $j \to \infty$.

Since the nonlinear mapping $G : C(\Gamma) \to C(\Gamma)$ is continuous, we have also by (4.3),

$$\lim_{j\to\infty} \|I_{h_j}G(S\phi_j) - G(Sv)\|_{L^q(\Gamma)} = 0.$$

Writing the identity $I_{h_j}A(\phi_j)=I_{h_j}f$ in the form

$$\phi_j=2I_{h_h}\{D^*\phi_j-G(S\phi_j)\}+2I_{h_j}f,$$

the right-hand side converges strongly in $L^q(\Gamma)$. Hence also the sequence ϕ_j converges strongly. Since the weak limit is unique, we finally obtain

$$v=2(D^*v-G(Sv))+2f,$$

which is equivalent to $A(v)=f$. □

Next we shall demonstrate that the linearized equations are uniquely solvable.

Lemma 4.2. *Let*

$$A'_h(\sigma)v:=\left(\tfrac{1}{2}I-I_hD^*\right)v+I_hG'(S\sigma)Sv$$

be the approximation of the Fréchet derivative $A'(\sigma)$. For sufficiently small $h>0$ it is an isomorphism in $L^q(\Gamma)$.

Proof. By Theorem 3.4 for fixed $\sigma\in L^q(\Gamma)$ there exists $C=C(\sigma)>0$ such that

$$\|A'(\sigma)v\|_{L^q(\Gamma)}\geqslant C(\sigma)\|v\|_{L^q(\Gamma)},\quad v\in L^q(\Gamma),\tag{4.4}$$

and $A'(\sigma):L^q(\Gamma)\to L^q(\Gamma)$ is an isomorphism. We shall show that for all $h\leqslant h_0$, where h_0 is sufficiently small, it holds

$$\|A'_h(\sigma)v\|_{L^q(\Gamma)}\geqslant \tfrac{1}{2}C(\sigma)\|v\|_{L^q(\Gamma)}.$$

For this we write

$$A'_h(\sigma)v=A'(\sigma)v+(I-I_h)D^*v-(I-I_h)G'(S\sigma)Sv.\tag{4.5}$$

First we note that $D^*:L^q(\Gamma)\to W^{1,q}(\Gamma)$ is a bounded operator by Theorem 3.1 (or [17]). Hence we have by the approximation properties of the interpolation operator,

$$\|(I-I_h)D^*v\|_{L^q(\Gamma)}\leqslant ch\|D^*v\|_{W^{1,q}(\Gamma)}\leqslant ch\|v\|_{L^q(\Gamma)}.$$

By the definition of the Fréchet derivative $G'(u)$ it is a multiplicative operator in $L^q(\Gamma)$. This means that for every $v\in L^q(\Gamma)$,

$$G'(u)v(x)=G'(u(x))v(x),\quad \text{for almost all } x\in\Gamma.$$

By Assumption A_3 the function $G'(u(\cdot))$ is continuous if $u\in C(\Gamma)$. Thus the restriction of the Fréchet derivative

$$G'(u(\cdot))|_{C(\Gamma)}:C(\Gamma)\to C(\Gamma)$$

is a bounded operator. Since the single-layer operator $S:L^q(\Gamma)\to C(\Gamma)$ is compact and $I-I_h:C(\Gamma)\to L^q(\Gamma)$ bounded, the operator

$$(I-I_h)G'(S\sigma)S:L^q(\Gamma)\to L^q(\Gamma)$$

is compact. By the approximation properties of the splines we know that for every $v\in L^q(\Gamma)$,

$$\|(I-I_h)G'(S\sigma)Sv\|_{L^q(\Gamma)}\leqslant c(\Gamma)\|(I-I_h)G'(S\sigma)Sv\|_{L^\infty(\Gamma)}\to 0,$$

as $h \to 0$. Using the results of [10, Section 8.5, p.186], we can conclude that

$$\|(I-I_h)G'(S\sigma)S\|_{L^q(\Gamma)\to L^q(\Gamma)} \leqslant \epsilon(h) \to 0,$$

as $h \to 0$.

Now we can choose $h_0 > 0$ such that for all $0 < h \leqslant h_0$,

$$\|(I-I_h)D^* v\|_{L^q(\Gamma)} \leqslant \tfrac{1}{4}C(\sigma)\|v\|_{L^q(\Gamma)}, \qquad \|(I-I_h)G'(S\sigma)Sv\|_{L^q(\Gamma)} \leqslant \tfrac{1}{4}C(\sigma)\|v\|_{L^q(\Gamma)}.$$

With the help of these estimates together with the triangle inequality and (4.4), we obtain

$$\|A'_h(\sigma)v\|_{L^q(\Gamma)} \geqslant c(u)\|v\|_{L^q(\Gamma)} - \tfrac{1}{2}c(u)\|v\|_{L^q(\Gamma)} \geqslant \tfrac{1}{2}c(u)\|v\|_{L^q(\Gamma)}.$$

This implies that $A'_h(\sigma): L^q(\Gamma) \to L^q(\Gamma)$ is an injective operator and bounded.

But since

$$\|A'_h(\sigma) - A'(\sigma)\|_{L^q(\Gamma)\to L^q(\Gamma)} \leqslant ch + \epsilon(h),$$

the operator

$$\left[I + [A'_h(\sigma) - A'(\sigma)]A'(\sigma)^{-1}\right]$$

is invertible for sufficiently small $h > 0$. Therefore also

$$A'_h(\sigma) = \left[I + [A'_h(\sigma) - A'(\sigma)]A'(\sigma)^{-1}\right]A'(\sigma)$$

is invertible. □

Next we shall show that the finite-dimensional equation

$$I_h A(u_h) = I_h f$$

has at least one solution.

We need the following lemma.

Lemma 4.3. *The degree of the mapping* $I_h A(\cdot)$:

$$d(I_h A(\cdot),\, B_h(u, r),\, I_h f) = \pm 1.$$

Proof. As a consequence of the previous lemma the finite-dimensional operator

$$I_h A'(\sigma)|_{S_N^d(\Gamma)} = A'_h(\sigma)|_{S_N^d(\Gamma)} : S_N^d(\Gamma) \to S_N^d(\Gamma)$$

is a homeomorphism. Hence the Brouwer degree of this mapping:

$$d(I_h A'(\sigma),\, B(0, r) \cap S_N^d(\Gamma),\, 0) = \pm 1,$$

for every $r > 0$.

Since $u \in L^q(\Gamma)$ is the only solution of the nonlinear integral equation

$$A(u) = \tfrac{1}{2}u - D^* u + G(Su) = f,$$

for every $v \in \{v \in L^q(\Gamma) \mid \|u - v\| = r\}$ it holds $A(v) - f \neq 0$, where $r > 0$ is arbitrary, but fixed.

Next we show that on the boundary of the set $B_h(u, r) = \{\phi \in S_N^d(\Theta) \mid \|\phi - u\| \leqslant r\}$ there are no solutions of the collocation equations.

Let us now fix some $r > 0$. Assume now, contrary to our statement, that we can find a sequence $\phi_j \in S^d_{N_j}(\theta)$, $\|u - \phi_j\| = r$, such that

$$I_{h_j} A(\phi_j) = I_{h_j} f.$$

Because A is a-proper with respect to the projection scheme $\pi = \{S^d_N(\theta), I_h\}$, there exists $v_0 \in L^q(\Gamma)$ such that $\lim_{j\to\infty} \|\phi_j - v_0\|_{L^q(\Gamma)} = 0$, $A(v_0) = f$ and $\|u - v_0\|_{L^q(\Gamma)} = r$. This is against the assumption that $u \in L^q(\Gamma)$ is the only solution.

Using the results of the the general-degree theory [8] we know that

$$d(I_h A(\cdot), B_h(u, r), I_h f) = d(I_h A'(\cdot), B_h(0, r), 0).$$

But the degree of the linear mapping $I_h A'(u)$ with respect to zero

$$d(I_h A'(u), B_h(0, r), 0) = \pm 1,$$

by Lemma 4.2. Thus the statement is proved. □

As a consequence from Lemma 4.3 we conclude that the collocation equations (4.2) admit a solution $u_h \in S^d_N(\theta)$. It remains to prove that the solutions are unique for every $h > 0$. First we shall show the following lemma.

Lemma 4.4. *The sequence of solutions $\{u_h\}$ to the collocation equations converges strongly in $L^q(\Gamma)$ to the solution u of the integral equation* (3.3).

Proof. Since all the solutions u_h of the collocation equations are bounded (Lemma 4.3), we may apply the a-properness of the nonlinear operator A. First we can only conclude that all subsequences converges to some solution of

$$\tfrac{1}{2}u - D^*u + G(Su) = f.$$

But the solution of this equation is unique. Hence we obtain the convergence of the sequence of collocation solutions. □

Next we shall prove the uniqueness.

Lemma 4.5. *For all $h \leqslant h_0$, where h_0 is sufficiently small, the solution $u_h \in S^d_{N(h)}(\theta)$ of the collocation equation* (4.2) *is unique.*

Proof. Since $A(\cdot)$ is continuously Fréchet differentiable, we can find $\rho > 0$ such that for every $u_h \in B(u, \rho) \cap S^d_N(\theta)$ there holds

$$\| I_h A'(u_h)\phi \|_{L^q(\Gamma)} \geqslant c(u) \| \phi \|_{L^q(\Gamma)}, \quad \phi \in S^d_N(\theta). \tag{4.6}$$

Assume now that u_h, $v_h \in S^d_n(\theta)$ are solutions of the collocation equation, i.e.,

$$I_h[A(u_h) - A(v_h)] = 0.$$

We choose $h_0 > 0$ so small that for every $0 < h \leqslant h$,

$$u_h, v_h \in B(u, \rho), \quad \|u_h - v_h\|_{L^q(\Gamma)} < \tfrac{1}{2}\epsilon. \tag{4.7}$$

Here $\frac{1}{2}\epsilon$ can be made so small that the remainder

$$R(u_h;\, u_h - v_h) < \tfrac{1}{2}c(u). \tag{4.8}$$

Combining the estimates (4.6)–(4.8) with

$$\| I_h[A(u_h) - A(v_h)] \|_{L^q(\Gamma)} \geqslant \| I_h DA(u_h)(u_h - v_h) \|_{L^q(\Gamma)} - R(u_h;\, u_h - v_h) \| u_h - v_h \|_{L^q(\Gamma)},$$

we finally get

$$0 \geqslant \tfrac{1}{2}c(u) \| u_h - v_h \|_{L^q(\Gamma)},$$

which proves the uniqueness. □

From the Lemmas 4.2–4.4 we finally conclude the main theorem.

Theorem 4.6. *There exists $h_0 > 0$ such that for all $0 < h < h_0$ the collocation equation*

$$I_h A(u_h) = I_h f$$

has a unique solution $u_h \in S^d_{N(h)}(\theta)$. Furthermore, the sequence of solutions converges strongly in $L^q(\Gamma)$ towards the solution of the nonlinear integral equation

$$A(u) = f.$$

The asymptotic error estimates follow essentially from the mapping properties of the integral operators and Assumption A_3, where we assumed that the nonlinearity is a continuously differentiable mapping and there holds for almost all $x \in \Gamma$,

$$| G(u)(x) - G(v)(x) | \leqslant \left(K_1 + K_2\left[| u(x) |^{p-2} + | v(x) |^{p-2} \right] \right) | u(x) - v(x) |.$$

Next we shall prove the asymptotic error estimates.

Theorem 4.7. *Let $u \in W^{s,q}(\Gamma)$ be the solution of the boundary integral equation $A(u) = f$. Then there exists $h_0 > 0$ such that for all $0 < h < h_0$ there holds*

$$\| u_h - u \|_{L^q(\Gamma)} \leqslant c h^s \| u \|_{W^{s,q}(\Gamma)},$$

where the constant $c = c(u)$ is independent of the mesh parameter h.

Proof. As in the proof of Lemma 4.2 there exists $h_0 > 0$ such that for all $\phi \in S^d_h(\Gamma)$,

$$\| I_h A'(I_h u)\phi \|_{L^q(\Gamma)} \geqslant c(u) \| \phi \|_{L^q(\Gamma)}, \tag{4.9}$$

where $0 < h < h_0$. This follows from the continuity of the Fréchet derivative $A'(\cdot)$.

By Assumption A_3 there exists a function $\epsilon(v, \psi) \in C(\Gamma)$ such that

$$\| \epsilon(v, \psi) \|_{L^\infty(\Gamma)} \to 0,$$

as $\psi \to 0$ in $C(\Gamma)$ for all $v \in C(\Gamma)$ and

$$G(v + \psi) - G(v) = G'(v)\psi + \epsilon(v, \psi) \| \psi \|_{L^\infty(\Gamma)}.$$

Furthermore, $\epsilon(\cdot,\ \cdot)$ is continuous with respect to the function v in the L^∞-topology. Thus we obtain

$$\begin{aligned} I_h(A(u_h)-A(I_hu)) &= \left[\tfrac{1}{2}I - I_hD^*\right]\phi + I_h\left[G(Su_h) - G(S(I_hu))\right] \\ &= \left[\tfrac{1}{2}I - I_hD^*\right]\phi + I_h\left[G'(S(I_hu))S\phi + I_h(\epsilon(S(I_hu),\ S\phi)\|S\phi\|_{L^\infty(\Gamma)}\right), \end{aligned} \tag{4.10}$$

where $\phi = u_h - I_hu \in S_h^d$. Since by the definition of the collocation solutions $I_hA(u_h) = I_hA(u)$, the left-hand side can be written as

$$I_h(A(u_h)-A(I_hu)) = -I_hD^*(u-I_hu) + I_h(G(Su) - G(S(I_hu))). \tag{4.11}$$

From the equations (4.10) and (4.11) we get

$$I_h\left[A'(I_hu)\phi + \epsilon(S(I_hu),\ S\phi)\|S\phi\|_{L^\infty(\Gamma)}\right] = I_h\left[D^*(u-I_hu) + G(Su) - G(S(I_hu))\right]. \tag{4.12}$$

Using the uniform boundedness of the interpolation operators $I_h : C(\Gamma) \to S_h^d$ and the compactness of the single-layer operator $S : L^q(\Gamma) \to C(\Gamma)$, we obtain

$$\| I_h[\epsilon(S(I_hu),\ S\phi)] \|_{L^q(\Gamma)} \| S\phi \|_{L^\infty(\Gamma)} \leqslant M\ \|\epsilon(S(I_hu),\ S\phi)\|_{L^\infty(\Gamma)} \|\phi\|_{L^q(\Gamma)}.$$

By Theorem 4.6 we have $u_h \to u$ in $L^q(\Gamma)$, and by the approximation properties, $I_hu \to u$ strongly in $L^q(\Gamma)$. Thus $S(I_hu - u) = S(\phi) \to 0$ in $C(\Gamma)$ as h tends to zero. Hence by the continuity of the derivative $G'(\cdot)$ in $C(\Gamma)$ we can choose $h_0 > 0$ such that

$$\|\epsilon(S(I_hu),\ S\phi)\|_{L^\infty(\Gamma)} \leqslant \frac{c(u)}{2M\ \|S\|_{L^q \to L^\infty}},$$

for all $0 < h < h_0$. This estimate together with (4.12) and the norm estimate (4.9) gives us by the triangle inequality

$$\tfrac{1}{2}c(u)\|\phi\|_{L^q(\Gamma)} \leqslant \| I_hD^*(u-I_hu)\|_{L^q(\Gamma)} + \| I_h[G(Su) - G(S(I_hu))] \|_{L^q(\Gamma)}.$$

The first term on the right-hand side can be estimated, using Theorem 3.1, to obtain

$$\| I_hD^*(u-u_h)\|_{L^q(\Gamma)} \leqslant c\|u - I_hu\|_{L^q(\Gamma)}.$$

To estimate the second term, we use the uniform boundedness of the interpolation operator, Assumption A_3 and Theorem 3.1, which gives us

$$\begin{aligned} \| I_h[G(Su) - G(S(I_hu))] \|_{L^q(\Gamma)} &\leqslant \|G(Su) - G(S(I_hu))\|_{L^\infty(\Gamma)} \\ &\leqslant \|Su - S(I_hu)\|_{L^\infty(\Gamma)}\left\{K_1 + K_2\left(\|Su\|_{L^\infty(\Gamma)}^{p-2} + \|S(I_hu)\|_{L^\infty(\Gamma)}^{p-2}\right)\right\} \\ &\leqslant \kappa(\|u\|_{L^q(\Gamma)})\|u - I_hu\|_{L^q(\Gamma)}. \end{aligned}$$

Combining the above estimates, we finally get

$$\tfrac{1}{2}c(u)\|u_h - I_hu\|_{L^q(\Gamma)} \leqslant \left[\kappa(\|u\|_{L^q(\Gamma)}) + c\right]\|u - I_hu\|_{L^q(\Gamma)}.$$

Table 1

Number of points	Relative error	Convergence rate
12	$1.9 \cdot 10^{-3}$	–
32	$8.4195 \cdot 10^{-5}$	3.1774
64	$9.4297 \cdot 10^{-6}$	3.1585

The statement of the theorem now follows by the approximation properties of the interpolation operators. □

Numerical example. Assume that Ω is a disc with a radius $a = \sqrt{e}$.The nonlinear problem is as follows:

$$\begin{cases} \Delta\Phi = 0, \\ -\dfrac{\partial\Phi}{\partial n}(x) = |\Phi(x)|\Phi(x)^3 - f(x), \end{cases}$$

where

$$f(x(t)) = \tfrac{1}{2}\cos(2t) + \tfrac{1}{4}a^4|\cos(2t)|\cos(2t)^3.$$

The exact solution is $u(x(t)) = \cos(2t)$. In Table 1 the experimental convergence rate is shown. All calculations are carried out using the PC-compatible desktop computer with Intel-486 processor using the PCMATLAB-software. The numerical examples demonstrate that the boundary element formulation is a very powerful method for solving this kind of nonlinear problems in engineering applications yielding extremely accurate results. We want to point out that the evaluation of the potential inside the domain can be done using the fast Poisson solver. But this is already beyond the scope of this paper. Finally we remark that the direct computation of the potential using the boundary integral representation will yield exactly the same convergence rate as for the boundary distribution.

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